MATH 2010B Advanced Calculus I Lecture Notes Week 9

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Last time ... Differentiability of $f(x,y)$ at (x_0, y_0)

Def$\underline{n}$: $f: \mathbb{R}^n \to \mathbb{R}^m$ is diff. at $\vec{x}_0 \in \mathbb{R}^n$

if $\exists$ linear transformation $Df(\vec{x}_0) : \mathbb{R}^n \to \mathbb{R}^m$

s.t. $$f(\vec{x}) = \underbrace{f(\vec{x}_0) + Df(\vec{x}_0)(\vec{x} - \vec{x}_0)}_{L(\vec{x})} + \underbrace{\varepsilon(\vec{x})}_{\text{error}} .$$

where $$\lim_{\vec{x} \to \vec{x}_0} \frac{\varepsilon(\vec{x})}{\|\vec{x} - \vec{x}_0\|} = 0 .$$

In matrix form, if $f = (f_1, \dots, f_m)$ in components

$$Df(\vec{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} (\vec{x}_0)$$

$m \times n$ matrix!

Summary Chart

$\boxed{f \in C^1 \text{ at } \vec{x}_0}$ $\Longrightarrow$ $\boxed{f \text{ is differentiable at } \vec{x}_0}$ $\Longrightarrow$ $\boxed{f_{x_1}, \dots, f_{x_n} \text{ exist at } \vec{x}_0}$

$\boxed{f \text{ is differentiable at } \vec{x}_0}$ $\Longrightarrow$ $\boxed{f \text{ is cts at } \vec{x}_0}$

$\boxed{f_{x_1}, \dots, f_{x_n} \text{ exist at } \vec{x}_0}$ $\not\Leftrightarrow$ $\boxed{f \text{ is cts at } \vec{x}_0}$

i.e. 1st order partial derivatives exist near $\vec{x}_0$ and is cts at $\vec{x}_0$

Chain Rule

(1D) $\quad y = f(x(t)) =: f \circ x(t)$

$$\Rightarrow \quad \underset{\text{at } t_0}{\frac{dy}{dt}} = \underset{\text{at } x(t_0)}{\frac{dy}{dx}} \cdot \underset{\text{at } t_0}{\frac{dx}{dt}}$$

General Chain Rule (Abstract)

Suppose we have two differentiable functions

$$\begin{array}{ccccc} \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^m & \xrightarrow{g} & \mathbb{R}^k \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \vec{x}_0 & \longmapsto & f(\vec{x}_0) & \longmapsto & g(f(\vec{x}_0)) \end{array}$$

Then, $g \circ f : \mathbb{R}^n \to \mathbb{R}^k$ is diff. at $\vec{x}_0$

and $\quad \underbrace{D(g \circ f)(\vec{x}_0)}_{\cancel{k\times m}\ k \times n \text{ matrix}} = \underbrace{Dg(f(\vec{x}_0))}_{k \times m \text{ matrix}} \cdot \underbrace{Df(\vec{x}_0)}_{m \times n \text{ matrix}}$.

Restrict to some simple cases

Case 1: $n = k = 1$, $m = 2$.

$$\begin{array}{ccccc} t & & (x,y) & & \\ \mathbb{R} & \longrightarrow & \mathbb{R}^2 & \xrightarrow{w} & \mathbb{R} \\ & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ t & \longmapsto & (x(t), y(t)) & & w(x,y) \\ & & & \longmapsto & w(x(t), y(t)) \end{array}$$

Claim: $\quad \frac{d}{dt} w(x(t), y(t)) = \left.\frac{\partial w}{\partial x}\right|_{(x(t),y(t))} \cdot \frac{dx}{dt} + \left.\frac{\partial w}{\partial y}\right|_{(x(t),y(t))} \cdot \frac{dy}{dt}$.

Notation: $\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}}$

Example: Consider $w(x,y) = e^{-x^2-y^2}$

and $\begin{cases} x = t \\ y = \sqrt{t} \end{cases}$

Calculate $\frac{d}{dt} w(x(t), y(t))$

(i) directly after substitution.

(ii) using the Chain Rule.

Sol: (i) $w(x(t), y(t)) = e^{-t^2-t}$ (function in t).

$$\frac{d}{dt} w(x(t), y(t)) = \frac{d}{dt}\left(e^{-t^2-t}\right)$$

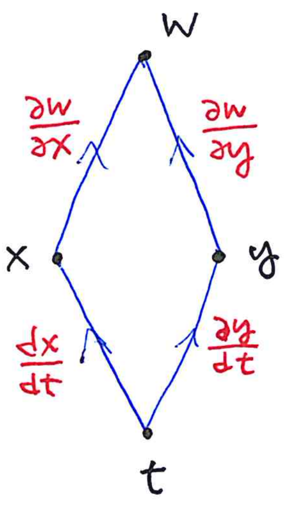
$$= e^{-t^2-t}(-2t-1) \quad \#.$$

(ii) Chain Rule: $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$.

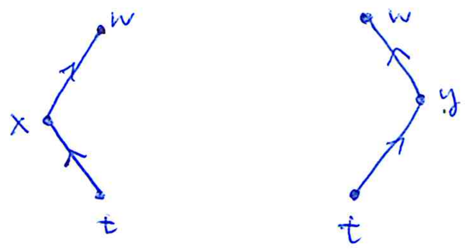
$$\frac{dw}{dt} = e^{-x^2-y^2}(-2x)\cdot(1) + e^{-x^2-y^2}(-2y)\left(\frac{1}{2\sqrt{t}}\right)$$

$$\begin{pmatrix} \text{Put} \\ x = t \\ y = \sqrt{t} \end{pmatrix} = e^{-t^2-t}\left[-2t-1\right] \quad \# \quad \text{(same as (i))}$$

Remember Chain Rule – "Branch diagram"



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

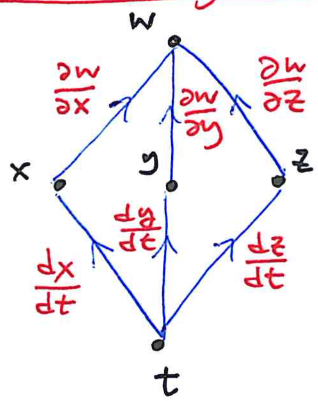


Case 2: $n = k = 1$, $m = 3$ $\mathbb{R} \longrightarrow \mathbb{R}^3 \longrightarrow \mathbb{R}$

E.g. Consider $W(x,y,z) = \sin(xyz)$

and $\begin{cases} x = t \\ y = t^2 \\ z = t^3 \end{cases}$

"Branch diagram"



evaluate $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$.

Sol: $\begin{cases} \frac{\partial w}{\partial x} = yz\cos xyz \\ \frac{\partial w}{\partial y} = xz\cos xyz \\ \frac{\partial w}{\partial z} = xy\cos xyz \end{cases}$; $\begin{cases} \frac{dx}{dt} = 1 \\ \frac{dy}{dt} = 2t \\ \frac{dz}{dt} = 3t^2 \end{cases}$

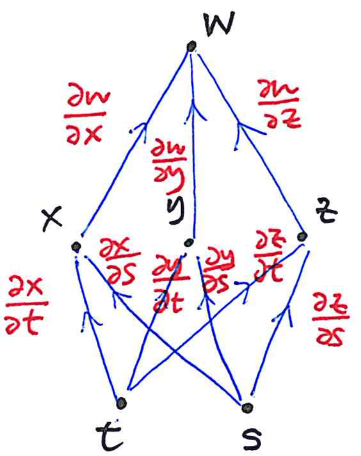
$$\Rightarrow \frac{dw}{dt} = \cos xyz \left[yz \cdot 1 + xz \cdot 2t + xy \cdot 3t^2 \right]$$

$$= (\cos t^6)(t^5 + 2t^5 + 3t^5)$$

$$= 6t^5 \cos t^6 \quad *.$$

Case 3: $n = 2$, $m = 3$, $k = 1$, $\mathbb{R}^2 \longrightarrow \mathbb{R}^3 \longrightarrow \mathbb{R}$

"Branch Diagram"



Chain Rule: $W = W(x,y,z) = W(x(t,s), y(t,s), z(t,s))$

function of t & s

$$\begin{cases} \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}. \\ \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \end{cases}$$

E.g. Consider $w(x,y,z) = \sqrt{x^2+y^2+z^2}$

and $\begin{cases} x = 3e^t \sin s \\ y = 3e^t \cos s \\ z = 4e^t \end{cases}$

Calculate $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$ at $t = s = 0$.

Sol: $\begin{cases} \frac{\partial w}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}} = 0 \\ \frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{3}{5} \\ \frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{4}{5} \end{cases}$ at $(x,y,z) = (0,3,4)$

[Note: $t = s = 0$
$\Rightarrow x = 0, y = 3, z = 4$]

$\begin{cases} \frac{\partial x}{\partial t} = 3e^t \sin s = 0 \\ \frac{\partial y}{\partial t} = 3e^t \cos s = 3 \\ \frac{\partial z}{\partial t} = 4e^t = 4 \end{cases}$ $\begin{cases} \frac{\partial x}{\partial s} = 3e^t \cos s = 3 \\ \frac{\partial y}{\partial s} = -3e^t \sin s = 0 \\ \frac{\partial z}{\partial s} = 0 = 0 \end{cases}$

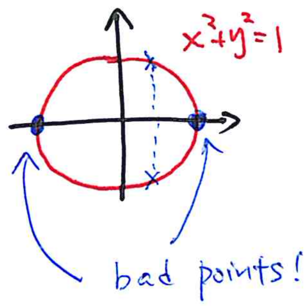
Therefore,

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} = 0 + 3\cdot\frac{3}{5} + 4\cdot\frac{4}{5} = 5.$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = 0 + 0 + 0 = 0.$$

$\blacksquare$

Applications of Chain Rule



(I) Implicit Differentiation.

$$f(x,y) = x^2+y^2 = 1 \quad \Rightarrow \quad y(x) = \pm\sqrt{1-x^2}$$

Go up 1-dimension, consider a level set

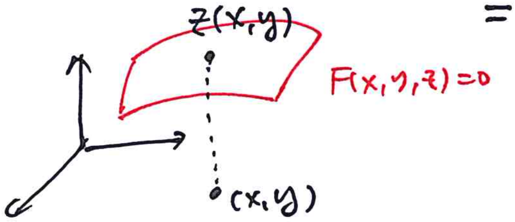
$$F(x,y,z) = 0 \quad \Rightarrow \quad \text{implicitly defines } z = z(x,y)$$

i.e. $F(x,y,z(x,y)) \equiv 0$ (identity)

Eg. $\underbrace{x+y+z}_{F(x,y,z)} = 0 \quad \Rightarrow \quad z(x,y) = -x-y.$

since $F(x,y,z(x,y))$

$= x+y+(-x-y) \equiv 0.$



Q: How to find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ if $z(x,y)$ is defined implicitly by the equation $F(x,y,z)=0$?

A: Just differentiate

Eg. Suppose $\boxed{x^3+y^3+z^3 = xyz}$ (*). Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

Take $\frac{\partial}{\partial x}$ of (*), Remember: only y is a constant. ($z=z(x,y)$)

$$3x^2 + 0 + 3z^2\frac{\partial z}{\partial x} = y\left(z + x\frac{\partial z}{\partial x}\right)$$

Then, solve for $\frac{\partial z}{\partial x}$.

$\boxed{\text{Ex: } \frac{\partial z}{\partial y} = ?}$

$$(3z^2 - xy)\frac{\partial z}{\partial x} = yz - 3x^2$$

$$\frac{\partial z}{\partial x} = \frac{yz-3x^2}{\underbrace{3z^2-xy}_{\text{draw back: in terms of } x,y \text{ AND } z}}.$$

General Formula: $F(x,y,z)=0$

implicitly $\Rightarrow$ $F(x,y,z(x,y))=0$

Take $\frac{\partial}{\partial x}$, chain rule

$$\frac{\partial F}{\partial x}\cancel{\frac{\partial x}{\partial x}}^{1} + \frac{\partial F}{\partial y}\cancel{\frac{\partial y}{\partial x}}^{0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \quad F_x + F_z \frac{\partial z}{\partial x} = 0$$

similarly,

$$\Rightarrow \quad \boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}} \qquad \boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

(II) Change of Variable.

E.g. $w=f(r)$, $r=\sqrt{x^2+y^2+z^2}$.

$w: \mathbb{R}^3 \to \mathbb{R}$ "rotationally symmetric".

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} = \frac{x}{r}\frac{\partial w}{\partial r}.$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{x}{r}\frac{\partial w}{\partial r}\right] = \frac{\partial}{\partial x}\left(\frac{x}{r}\right)\frac{\partial w}{\partial r} + \frac{x}{r}\left[\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right)\right]\left[\frac{\partial r}{\partial x}\right]$$

$$= \frac{r - x\frac{x}{r}}{r^2}\frac{\partial w}{\partial r} + \frac{x^2}{r^2}\frac{\partial^2 w}{\partial r^2}.$$

$$= \frac{r^2-x^2}{r^3}\frac{\partial w}{\partial r} + \frac{x^2}{r^2}\frac{\partial^2 w}{\partial r^2}.$$

[Ex: $\underbrace{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}}_{\Delta w} = \frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r}.$]